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Gravitational F-terms through anomaly equations and deformed chiral rings

Luis F. Alday,[ab] Michele Cirafici,[b] Justin R. David,[a] Edi Gava[ab] and Kumar S. Narain[a]

[a] *High Energy Section, The Abdus Salam International Centre for Theoretical Physics Strada Costiera, 11-34014 Trieste, Italy*
[b] *Instituto Nazionale di Fisica Nucleare, sez. di Trieste and SISSA Trieste, Italy*
E-mail: alday@he.sissa.it, cirafici@sissa.it, justin@ictp.trieste.it, gava@ictp.trieste.it, narain@ictp.trieste.it

Abstract: We study effective gravitational F-terms, obtained by integrating an $\mathrm{U}(N)$ adjoint chiral superfield Φ coupled to the $\mathcal{N}=1$ gauge chiral superfield W_α and supergravity, to arbitrary orders in the gravitational background. The latter includes in addition to the $\mathcal{N}=1$ Weyl superfield $G_{\alpha\beta\gamma}$, the self-dual graviphoton field strength $F_{\alpha\beta}$ of the parent, broken $\mathcal{N}=2$ theory. We first study the chiral ring relations resulting from the above non-standard gravitational background and find agreement, for gauge invariant operators, with those obtained from the dual closed string side via Bianchi identities for $\mathcal{N}=2$ supergravity coupled to vector multiplets. We then derive generalized anomaly equations for connected correlators on the gauge theory side, which allow us to solve for the basic one-point function $\langle \mathrm{Tr} W^2/(z-\Phi)\rangle$ to all orders in F^2. By generalizing the matrix model loop equation to the generating functional of connected correlators of resolvents, we prove that the gauge theory result coincides with the genus expansion of the associated matrix model, after identifying the expansion parameters on the two sides.

Keywords: M(atrix) Theories, Anomalies in Field and String Theories, Supersymmetric Effective Theories.

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Contents

1. Introduction

The issue of effective gravitational F-terms in $\mathcal{N}=1$ gauge theories coupled to $\mathcal{N}=1$ supergravity, generated by integrating out a massive $\mathrm{U}(N)$ adjoint superfield Φ, has received much attention recently [1]–[6], after the conjecture of [1], which has identified these terms to higher genus contributions to the free energy of a related hermitean matrix model, with potential given by the superpotential $W(\Phi)$ of the adjoint superfield. The conjecture was proved in [4, 5] using the diagrammatic techniques developed in [7] to prove the relation between the effective superpotential of the gauge theory and the planar, genus zero, contribution to the matrix model free energy. One of the ingredients of the proof in [4, 5] was the modification of the chiral ring of the gauge theory in the presence of a supergravitational background. The modification has two sources. The first can be understood within the framework of ordinary supergravity tensor calculus and related Bianchi identities in the presence of the $\mathcal{N}=1$ Weyl superfield $G_{\alpha\beta\gamma}$, and it reads $\{W_\alpha, W_\beta\} = 2G_{\alpha\beta\gamma}W^\gamma$, where W_α is the gauge chiral superfield. It turns out however that, in order to capture genus $g \geq 2$ contributions, in the language of the dual closed string theory side [8], one needs to introduce a more drastic modification in the chiral ring relation, to account for a non-trivial vacuum expectation value of the (self-dual) graviphoton field strength $F_{\alpha\beta}$ of the parent

$\mathcal{N} = 2$ string theory. In this case the relation reads $\{W_\alpha, W_\beta\} = F_{\alpha\beta} + 2G_{\alpha\beta\gamma}W^\gamma$. In particular, one is modifying the Grassmann nature of the fermionic superfield W_α. In [6] we considered the first kind of modification, involving only the Weyl superfield $G_{\alpha\beta\gamma}$, exploited its consequences and used generalized anomaly equations [9] to prove that, indeed, the order G^2 effective superpotential is given, as a function of $S = -\operatorname{Tr} W^2/32\pi^2$, by the genus one contribution to the free energy of the related matrix model. A crucial fact in that approach was the lack of factorization of chiral correlators in the presence of a non-trivial supergravity background, in particular the non-triviality of connected two-point correlators.

The purpose of the present work is to complete and extend the above analysis to arbitrary orders in F^2, i.e. to arbitrary genera on the matrix model side, or on the dual closed string side. This amounts, among other things, to determine the modifications in the chiral ring relations due to the presence of the $F_{\alpha\beta}$ background. It turns out that in this case one has to face ordering ambiguities in manipulating W's in the generalized Konishi anomaly equations, somewhat similar to the base point dependence in path ordered exponentials found in [4, 5]. We fix the ambiguities by requiring that traces of (graded) commutators be trivial in the chiral ring. This requirement will lead us to get identities involving gauge invariant and gravitational operators, which will be very important when analyzing the anomaly equations. While we do not have a proof of this assumption at the gauge theory level, quite remarkably we will find that these relations are exactly the same as those we obtain on the dual closed string theory side by using the Bianchi identities of the (broken) $\mathcal{N} = 2$ supergravity theory coupled to vector multiplets. We will then first, following the strategy of [6], reconsider the $g = 1$ case in the presence of both $F_{\alpha\beta}$ and $G_{\alpha\beta\gamma}$ and find that the order F^2 and G^2 superpotential terms have the expected structure and agree with the $g = 1$ matrix model result.

In order to proceed with arbitrary genus analysis, for which a non-trivial $F_{\alpha\beta}$ is essential, we will need to further generalize the strategy of [6] to derive anomaly equations for the generating functional of (connected) correlators, by coupling the relevant chiral gauge invariant operators to external sources. From these equations we will extract the correlators which are required to determine the one-point functions of $R(z)$, $T(z)$ and $w_\alpha(z)$ (in the notation of [9] reviewed in section 2) to all orders in F^2. From these one can determine the effective superpotential as explained in [9]. The connection with the matrix model will be proved by generalizing the loop equation of the latter to a full generating functional of connected correlators of resolvents, by coupling the matrix model resolvent $\Omega_m(z)$ to an external source. We will prove that, in fact, the gauge theory $\langle R(z)\rangle$ coincides with the matrix model $\langle \Omega_m(z)\rangle$, to all orders, if we identify F^2 with the matrix model genus-expansion parameter $1/\hat{N}^2$. More precisely, there is a full class of gauge theory correlators, $\langle \operatorname{Tr} W^2\Phi^k\rangle$, whose expansion in powers of F^2 coincides with the $1/\hat{N}^2$ expansion of $\langle \operatorname{Tr} M^k\rangle$, where M is a $\hat{N} \times \hat{N}$ hermitean random matrix, whose expectation value is computed with the measure $\exp(-\frac{\hat{N}}{g_m} \operatorname{Tr} W(M))$. We find this fact quite remarkable.

It is also worth mentioning that lower genus contributions to a given genus term can be gotten rid of by an operator redefinition, $R \to R + \frac{1}{6}G^2T$, thereby generalizing the shift $S \to S + \frac{1}{6}G^2N$ needed to remove the genus zero contribution to the order G^2 term.

It is interesting to note that, by following [9] and introducing an auxiliary Grassman coordinate ψ_α, which can be thought of as a second supercoordinate of the broken $\mathcal{N} = 2$ parent theory, we can rewrite the anomaly equations in a shift-invariant way, if, in addition to assembling R, w_α and T in $\mathcal{R}(\psi)$, as in [9], we assemble also F and G as $\mathcal{H} = F_{\alpha\beta} - \frac{1}{2}\psi^\gamma G_{\alpha\beta\gamma}$. This suggests that the effective superpotential can be formally written in a manifest shift-invariant way, $\int d^2\psi \mathcal{H}^{2g}\mathcal{F}_g(S + \psi^\alpha w_\alpha - \frac{1}{2}\psi^2 N)$. However, it will turn out that, due to the chiral ring relations, to be derived in section 2, the $g \geq 2$ terms are all trivial from the $\mathcal{N} = 1$ point of view. We should stress, nevertheless once again, that the all-orders identification of the gauge theory $\langle R(z)\rangle$ with the matrix model $\langle\Omega_m(z)\rangle$, implying the exact identity of an infinite family of correlators on the two sides, survives the chiral ring relations, since on the gauge theory side it involves powers of F^2 only.

The paper is organized as follows: in section 2 we discuss the chiral ring relations in the gauge theory with background F and G and those coming from the dual closed string side. In section 3 we consider the genus one contribution, extending the discussion of [6] to the case where both F and G are non-trivial and proving the expected structure of order G^2 and F^2 terms. In section 4 we derive anomaly equations for the generating functional of connected correlators, find the relevant connected two-point functions and finally the all orders one point function of T, from which the superpotential can be integrated. We also verify the shift symmetry of our expressions and prove agreement with the matrix model result. In appendix A we prove the consistency of the anomaly equations for the generating functional.

2. The chiral ring

The chiral ring in $\mathcal{N} = 1$ gauge theories consists of all operators which are annihilated by the covariant derivative $\bar{D}_{\dot\alpha}$, which is conjugate to the supercharge $\bar{Q}_{\dot\alpha}$, modulo $\bar{D}_{\dot\alpha}$ exact terms, where the exact terms should be gauge invariant and local operators. All relations in the chiral ring are therefore defined modulo $\bar{D}_{\dot\alpha}$ exact terms. The chiral ring and the various relations in it play an important role in deriving the effective F-terms in $\mathcal{N} = 1$ gauge theory obtained by integrating out the adjoint matter [9]. This is because the contributions to the effective action of all $\bar{D}_{\dot\alpha}$ exact terms can be written as an integral involving both holomorphic and anti-holomorphic integrations in superspace, $\int d^2x d^2\theta d^2\bar\theta S(\theta, \bar\theta)$. Thus they do not contribute to the F-terms, which involve only either a holomorphic or anti-holomorphic integrations in the superspace co-ordinates.

For the $\mathcal{N} = 1$ gauge theory with a single adjoint field on R^4 the chiral ring relations are given by [9][1]

$$[W_\alpha, \Phi] = 0 \bmod \bar{D}\,, \qquad \{W_\alpha, W_\beta\} = 0 \bmod \bar{D}\,, \tag{2.1}$$

where W_α is the $\mathcal{N} = 1$ gauge chiral superfield and Φ is the chiral matter superfield in the adjoint representation of the gauge group. In this paper we are interested in studying the

[1] All relations in the chiral ring are modulo $\bar{D}$, for the rest of paper this will not be explicitly mentioned, but understood wherever necessary.

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F-terms in $\mathcal{N}=1$ U(N) gauge theory obtained by integrating out a single adjoint scalar in presence of gravity as well as the self-dual graviphoton field strength $F_{\alpha\beta}$, using the generalized anomaly equations approach. The chiral ring relations in the presence of these backgrounds are given by [4, 5]

$$[W_\alpha, \Phi] = 0\,, \qquad \{W_\alpha, W_\beta\} = F_{\alpha\beta} + 2G_{\alpha\beta\gamma}W^\gamma\,. \tag{2.2}$$

In the presence of just a curved background the ring relations can be derived by using the Bianchi identities of $\mathcal{N}=1$ gravity, but for obtaining the ring relation in the presence of the graviphoton one has to resort to string theory. It was pointed out in [4] that the modification of ring relations in the presence of $F_{\alpha\beta}$ requires a non-traditional interpretation. The classical grassmanian nature of the $\mathcal{N}=1$ gauge multiplet no longer holds, the gauge multiplet in fact satisfies a Clifford algebra. All the gauge invariant operators constructed out of the basic fields of the $\mathcal{N}=1$ gauge multiplet and the chiral multiplet can be arranged into the following operators

$$\begin{aligned}
\mathcal{R}(z)_{ij} &= -\frac{1}{32\pi^2}\left(\frac{W^2}{z-\Phi}\right)_{ij}, & R(z) &= \operatorname{Tr}\mathcal{R}(z)\,,\\
\rho_\alpha(z)_{ij} &= \frac{1}{4\pi}\left(\frac{W_\alpha}{z-\Phi}\right)_{ij}, & w_\alpha(z) &= \operatorname{Tr}\rho_\alpha(z)\,,\\
\mathcal{T}(z)_{ij} &= \left(\frac{1}{z-\Phi}\right)_{ij}, & T(z) &= \operatorname{Tr}\mathcal{T}(z)\,.
\end{aligned} \tag{2.3}$$

Here, for later convenience, we have defined separate symbols for the matrix elements operators and the gauge invariant, trace operators. Placing more W's in the trace does not yield any more gauge invariant operators, as they can be converted to one of the above operators by the ring relations in (2.2), therefore the above set of operators is exhaustive in the chiral ring. In the next sub-section we will derive various identities from the relations (2.2), with a motivated ansatz that the adjoint action with W_α on gauge invariant operators vanishes. In the subsequent sub-section we will re derive these identities from the closed string dual using the $\mathcal{N}=2$ Bianchi identities, thus justifying the ansatz.

2.1 Chiral ring identities from gauge theory

From the definition of W^2 and the basic ring equations in (2.2) we can derive the following identities using simple algebraic manipulations,

$$\begin{aligned}
[W_\alpha, W^2] &= -2F_{\alpha\beta}W^\beta\,,\\
\{W_\alpha, W^2\} &= -\frac{2}{3}\left(G^2 W_\alpha + G_{\alpha\beta\gamma}F^{\beta\gamma}\right),
\end{aligned} \tag{2.4}$$

adding the above equations we find

$$W_\alpha W^2 = -F_{\alpha\beta}W^\beta - \frac{1}{3}G^2 W_\alpha - \frac{1}{3}G_{\alpha\beta\gamma}F^{\beta\gamma}\,. \tag{2.5}$$

Considering the equation (2.2) with a product of arbitrary number of scalars Φ and taking trace on both sides of the equation, we obtain

$$\begin{aligned}
\operatorname{Tr}(\{W_\alpha, W_\beta\}\Phi\Phi\cdots) &= \operatorname{Tr}((F_{\alpha\beta} + 2G_{\alpha\beta\gamma}W^\gamma)\Phi\Phi\cdots)\,,\\
&= \operatorname{Tr}(\mathrm{Ad}_{W_\alpha}W_\beta\Phi\Phi\cdots)\,.
\end{aligned} \tag{2.6}$$

We see that on the left hand side of the equation we have the adjoint action on the operator $W_\beta\Phi\Phi\ldots$. For ordinary Grassman W_α, the trace of the adjoint action is zero, but here, from the algebra in (2.2), it is not clear that this will still hold. However if the trace of the adjoint action is not zero, then in any gauge invariant operator, like the one considered in (2.6), there will be an ambiguity in the ordering of W_α, such that the cyclic property of the trace will not be obeyed. This is the base point ambiguity noted in [4]. To remove such ambiguities we demand that the trace of the adjoint action is trivial in the chiral ring. This leads us to the following equation

$$\mathrm{Tr}((F_{\alpha\beta} + 2G_{\alpha\beta\gamma}W^\gamma)\Phi\cdots) = 0\,. \tag{2.7}$$

We can write the above equation compactly in terms of the operators in (2.3). After performing the following convenient redefinitions, $F_{\alpha\beta} \to 32\pi^2 2\sqrt{2}F_{\alpha\beta}$ and $G_{\alpha\beta\gamma} \to \sqrt{32\pi^2}\, G_{\alpha\beta\gamma}$, (2.7) can be written as [2]

$$2F_{\alpha\beta}T + G_{\alpha\beta\gamma}w^\gamma = 0\,. \tag{2.8}$$

A similar equation can be obtained by considering the first equation in (2.4) with products of Φ's and a trace on both sides, we obtain

$$\mathrm{Tr}([W_\alpha, W^2]\Phi\cdots) = -2F_{\alpha\beta}\,\mathrm{Tr}(W^\beta\Phi\cdots)\,. \tag{2.9}$$

Again demanding that the adjoint action is trivial in the chiral ring we get

$$F_{\alpha\beta}\,\mathrm{Tr}(W^\beta\Phi\cdots) = 0\,, \tag{2.10}$$

that written in terms of the operators in (2.3) becomes

$$F_{\alpha\beta}w^\beta = 0\,. \tag{2.11}$$

Using Bianchi identities of $\mathcal{N} = 1$ supergravity it was shown in [6], that the spin 2 combination of a product of two $\mathcal{N} = 1$ Weyl multiplet was trivial in the ring. This combination is given by

$$G_{\alpha\beta\sigma}G^\sigma{}_{\gamma\delta} + G_{\alpha\gamma\sigma}G^\sigma{}_{\beta\delta} + G_{\alpha\delta\sigma}G^\sigma{}_{\beta\gamma} = 0\,. \tag{2.12}$$

This ensures that the gravitational corrections to the F-terms truncate at order G^2. If a similar Bianchi identity were applied on the symmetric tensor $F_{\alpha\beta}$ we would obtain the following product of the graviphoton and the $\mathcal{N} = 1$ Weyl multiplet

$$G_{\alpha\beta\gamma}F^\gamma{}_\sigma + G_{\alpha\sigma\gamma}F^\gamma{}_\beta\,, \tag{2.13}$$

which should therefore vanish in the chiral ring. However, a consistent $\mathcal{N} = 1$ gauge theory coupled to the spin 3/2 multiplet containing the graviphoton along with its supersymmetric partner, the gravitino, is at present not known. Such theories are consistent from string theory point of view as these are theories on D-branes with the graviphoton field strength

[2] From now on we will use these scaled variables for the rest of the paper.

turned on in the bulk. The presence of the graviphoton field strength would perhaps modify the Bianchi identity in (2.13), without an explicit construction of the gauge theory coupled to graviphoton it is not possible to determine the modification of the Bianchi identity on the $F_{\alpha\beta}$. The product of G and F in (2.13) contains both a spin 3/2 and a spin 1/2 part, but we make the minimal ansatz that the spin 3/2 combination in the tensor product is trivial in the chiral ring as given below

$$G_{\alpha\beta\gamma}F^{\gamma}_{\ \sigma} + G_{\alpha\sigma\gamma}F^{\gamma}_{\ \beta} + G_{\sigma\beta\gamma}F^{\gamma}_{\ \alpha} = 0\,. \tag{2.14}$$

From the basic identities in the chiral ring, (2.8), (2.11), (2.12) and (2.14), we derive other identities which are used at several instances in this paper. The first identity is obtained by multiplying the equation (2.11) by $F_{\alpha'}{}^{\alpha}$, which is gives

$$F_{\alpha'}{}^{\alpha}F_{\alpha\beta}w^{\beta} = \frac{1}{2}\epsilon_{\alpha'\beta}F^{\gamma\delta}F_{\gamma\delta}w^{\beta} = \frac{1}{2}F^2 w_{\alpha'} = 0\,. \tag{2.15}$$

Another important identity is given by

$$G^2 F_{\alpha\beta} = 0\,. \tag{2.16}$$

In order to prove this identity, we first multiply (2.14) by $G_{\delta}{}^{\alpha\beta}$ to obtain

$$\frac{1}{2}G^2 F_{\delta\sigma} - 2G_{\delta\beta\alpha}G^{\alpha}_{\ \sigma\gamma}F^{\gamma\beta} = 0\,. \tag{2.17}$$

The product $G_{\delta\beta\alpha}G^{\alpha}_{\ \sigma\gamma}$ contains both the spin 0 and the spin 2 part. The spin 2 part vanishes by (2.12), therefore this product contains only the spin 0 part, which is given by

$$G_{\delta\beta\alpha}G^{\alpha}_{\ \sigma\gamma} = -\frac{1}{6}\left(\epsilon_{\delta\sigma}\epsilon_{\beta\gamma} + \epsilon_{\beta\sigma}\epsilon_{\delta\gamma}\right)G^2\,. \tag{2.18}$$

Substituting the above equation in (2.17) gives (2.16). We also have the identity

$$F_{\alpha\beta}G^{\alpha\beta}{}_{\gamma}F^{\gamma\sigma} \equiv (F\cdot G)_{\gamma}F^{\gamma\sigma} = 0\,. \tag{2.19}$$

This equation is obtained by simply multiplying (2.14) by $F_{\alpha\beta}$, the last two terms vanish due to the symmetry of $G_{\alpha\beta\gamma}$ in all the indices. Finally, using (2.8) and (2.11) we can show that

$$(F\cdot G)_{\alpha}w_{\beta} = -(F\cdot G)_{\beta}w_{\alpha}\,. \tag{2.20}$$

The identities (2.15), (2.16), (2.18) and (2.19) imply that terms containing G^2 do not admit any expansions in higher powers of either F or G. And the terms containing $(F\cdot G)$ also do not admit any higher powers of either F or G. From these considerations we can conclude that the only expansion which admits arbitrary powers is an expansion purely in F.

2.2 Chiral ring identities from closed string dual

Though we do not have understanding of the chiral ring identities discussed in the previous subsection we can appeal to the open/closed string duality conjectured in [8] to derive them on the closed string side. The basic conjecture of [8]is that $\mathcal{N} = 1$, U(N) gauge theory with a single adjoint chiral superfield Φ, realized, for instance, by N D5-branes wrapped on a two-cycle of a local Calabi Yau threefold, is dual to closed string theory on a CY threefold which is related by a conifold transition to the previous one. On the closed string side $\mathcal{N}=2$ supersymmetry is broken down to $\mathcal{N}=1$ by the presence of three-form fluxes on the CY space [10]. The tree level superpotential for Φ is related to the geometry of the CY manifold. More precisely, the duality identifies the lowest moment of the gauge theory operators of (2.3) with the components of an U(1) $\mathcal{N}=2$ vector multiplet on the closed string side

$$V(\hat{\theta},\theta)=S(\theta)+\hat{\theta}^{\alpha}w_{\alpha}(\theta)+\hat{\theta}^2 N\,. \tag{2.21}$$

S is the closed string field dual to $\mathrm{Tr}(W^{\alpha}W_{\alpha})/32\pi^2$, and it corresponds to the complex structure modulus of the CY threefold, w_{α} is dual to $\mathrm{Tr}(W_{\alpha})/4\pi$ and N, the auxiliary field corresponding to the three-form flux on the closed string side, is dual to Tr(1) of the gauge theory. θ is the usual $\mathcal{N}=1$ superspace coordinate and $\hat{\theta}$ is the additional superspace coordinate for $\mathcal{N}=2$ superspace. The duality is expected to still hold after coupling the gauge theory on one side and the vector multiplet on the other side, to the supergravity background given by $F_{\alpha\beta}$ and $G_{\alpha\beta\gamma}$. In particular, a class of gravitational F-terms, usually called $\mathcal{F}_g$ [11, 12] are expected to match on the two sides, with the above identification of fields.

We can also organize the background fields, the $\mathcal{N}=1$ Weyl multiplet and the graviphoton field strength as an $\mathcal{N}=2$ Weyl multiplet as follows.

$$H_{\alpha\beta}(\hat{\theta},\theta)=F_{\alpha\beta}(\theta)+\hat{\theta}^{\gamma}G_{\alpha\beta\gamma}(\theta)\,. \tag{2.22}$$

In the above equation $F_{\alpha\beta}$ stands for the $\mathcal{N}=1$ self-dual graviphoton multiplet and $G_{\alpha\beta\gamma}$ refers to the $\mathcal{N}=1$ Weyl multiplet which contains the self-dual part of the Riemann curvature. We have set the auxiliary field of the Weyl multiplet to be zero as it does not play any role in deriving the ring relations discussed in the previous sub-section.

Our strategy to prove the basic ring relations in (2.8), (2.11) and (2.14) would be to use the $\mathcal{N}=2$ Bianchi identities to show that these equations are $\bar{D}$ exact. Here $\bar{D}$ refers to the derivative of the $\mathcal{N}=1$ superspace coordinate. This is the same method used to obtain the ring relations in (2.1). Consider the following $\bar{D}$ exact quantity

$$\bar{D}^{\dot{\alpha}}(D_{\alpha\dot{\alpha}}V)=[\bar{D}^{\dot{\alpha}},D_{\alpha\dot{\alpha}}]V\,. \tag{2.23}$$

In writing the equality we have used the fact that V, the $\mathcal{N}=2$ vector multiplet, is annihilated by $\bar{D}$. From the definition of covariant derivatives in superspace we have the following (see for instance [13])

$$\begin{aligned}&(\mathcal{D}_C\mathcal{D}_B-(-1)^{bc}\mathcal{D}_B\mathcal{D}_C)V^{(A_1A_2\cdots)}=\\&=-R_{CBD}^{A_1}V^{(DA_2\cdots)}-R_{CBD}^{A_2}V^{(A_1D\cdots)}-\cdots-T_{CB}^{D}\mathcal{D}_DV^{(A_1A_2\cdots)}\,.\end{aligned} \tag{2.24}$$

Here A,B,C,D, etc. refer either to bosonic or fermionic coordinates in superspace, b,c refer to their grading, a bosonic coordinate having grade 0 and a fermionic one grade 1. R and T stand for the curvature and the torsion in superspace coordinates respectively. For

$\mathcal{N}=2$ superspace, the complete solution for the Bianchi identities has been given in [14] and one can read out the required curvature and torsion symbols. The equation in (2.23) can then be written as

$$\bar{D}^{\dot{\alpha}}(D_{\alpha\dot{\alpha}}V) = T^{\dot{\alpha}}_{\alpha\dot{\alpha}\gamma}\hat{D}^{\gamma}V = H_{\alpha\gamma}\hat{D}^{\gamma}V\,. \tag{2.25}$$

There are no curvature contributions as V is a scalar in $\mathcal{N}=2$ superspace, there are other contributions to this Bianchi identity, but they vanish on shell.[3] The zeroth and the first component in $\hat{\theta}$ reduces to

$$\begin{aligned} F_{\alpha\gamma}w^{\gamma} &= 0\,, \\ 2F_{\alpha\beta}N + G_{\alpha\beta\gamma}w^{\gamma} &= 0\,. \end{aligned} \tag{2.26}$$

The $\hat{\theta}^2$ component is identically zero. These operator equations verify the lowest moment of the ring relations in (2.8) and (2.11) from the closed string side. To verify all the moments of these relations we need map which relates all the gauge invariant operators of the gauge theory to closed string fields, at present such a detail map is lacking, though it is obvious they will all be mapped to vector multiplets on the closed string side. To prove the relation in (2.14) consider the following $\bar{D}$ exact quantity

$$\begin{aligned} \bar{D}^{\dot{\alpha}}(D_{\alpha\dot{\alpha}}H^{\beta\gamma}) &= [\bar{D}^{\dot{\alpha}}, D_{\alpha\dot{\alpha}}]H^{\beta\gamma}\,, \\ &= R^{\dot{\alpha}\beta}{}_{\alpha\dot{\alpha}\rho}H^{\rho\gamma} + R^{\dot{\alpha}\gamma}{}_{\alpha\dot{\alpha}\rho}H^{\beta\rho} + T^{\dot{\alpha}}_{\alpha\dot{\alpha}\rho}\hat{D}^{\rho}H^{\beta\gamma}\,. \end{aligned} \tag{2.27}$$

Substituting the required curvature and torsion symbols we get [4]

$$\bar{D}^{\dot{\alpha}}(D_{\alpha\dot{\alpha}}H^{\beta\gamma}) = G^{\beta}{}_{\alpha\rho}H^{\rho\gamma} + G^{\gamma}{}_{\alpha\rho}H^{\rho\beta} + H_{\alpha\rho}\hat{D}^{\rho}H^{\beta\gamma}\,. \tag{2.28}$$

There are other terms in this Bianchi identity, but they all vanish on shell. From the lowest component in $\hat{\theta}$ of the above equation we obtain the relation (2.14). Note that on lowering the β and γ indices, the combination is entirely symmetric, thus containing only the spin $3/2$ part of the tensor product of G and F. This completes the proof of (2.14) from the closed string side. The linear term in $\hat{\theta}$ of (2.28) reduces to (2.12)

3. Genus one analysis

The chiral ring relations (2.1) ensures that only planar graphs contribute to the computation of the superpotential in the absence of gravity or the graviphoton field strength. The diagrammatic analysis of [4] and [5] show that higher genus diagrams contribute when either gravity or the graviphoton background is turned on. In fact the contribution of gravity alone enters at genus one in the superpotential, and it was shown in [6] that the genus one correction to the loop equation in the corresponding matrix model agrees with the gravitational corrected anomaly equations in the gauge theory. In this paper we extend

[3] This is the holomorphic counterpart in [14, equation (7.6)].

[4] We have used the holomorphic counterpart of [14, equations (4.25) and (7.6)].

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this to the situation when the graviphoton field strength is also turned on. The graviphoton affects the gauge theory loop equations at all genera. In this section as an important preliminary step to the analysis at all genera and, as a means to introduce all the definitions and methods, we will analyze the anomaly equations of the gauge theory with the graviphoton also turned on at genus one.

To derive the Ward identities constraining the gauge invariant generating functions of (2.3) in the presence of gravity and the graviphoton field strength we need three ingredients. Firstly, the background modifies the ring to (2.2) and the associated ring equations discussed in section 2.1 play a crucial role. The generalized Konishi anomaly [15] forms the second ingredient: one can derive the Ward identities constraining the functions $R(z)$, $w_\alpha(z)$ and $T(z)$ by considering an infinitesimal variation $\delta\Phi_{ij} = f_{ij}$ where f_{ij} is the matrix elements of the operators given in (2.3). This variation is anomalous and, in absence of gravity, the anomaly is given by

$$\frac{\delta f_{ji}}{\delta \Phi_{k\ell}} A_{ij,k\ell}\,, \tag{3.1}$$

with

$$A_{ij,k\ell} = (W^2)_{kj}\delta_{i\ell} + \delta_{kj}(W^2)_{i\ell} - 2W^\alpha_{kj}W_{\alpha i\ell}\,. \tag{3.2}$$

When a gravitational background is turned on, there is a direct contribution to the Konishi anomaly This is just the generalized gravitational contribution of the chiral anomaly [16, 17]. To include this contribution we replace $A_{ij,kl}$ in (3.2) with

$$A_{ij,k\ell} \to A_{ij,k\ell} + \frac{1}{3}G^2\delta_{kj}\delta_{i\ell}\,. \tag{3.3}$$

We expect that the presence of a graviphoton background will not affect the Konishi anomaly. The graviphoton field strength is of dimension 3, all terms in the Konishi anomaly equation are Lorentz scalars and of dimension 3. Thus there is no Lorentz invariant term which can be constructed out of the graviphoton field strength which is of dimension 3. Therefore, the effect of the graviphoton in the anomaly equations can be seen only through the ring (2.2). Using these two ingredients, the equations determining the gauge invariant operators of (2.3) are given by

$$\langle R(z)R(z)\rangle + \frac{1}{6}G^2\langle w^\alpha(z)w_\alpha(z)\rangle - \langle \mathrm{Tr}(V'(\Phi)\mathcal{R}(z))\rangle = 0\,, \tag{3.4}$$

$$2\langle R(z)w_\alpha(z)\rangle - \frac{1}{3}G^2\langle w_\alpha(z)T(z)\rangle - \langle \mathrm{Tr}(V'(\Phi)\rho_\alpha(z))\rangle = 0\,, \tag{3.5}$$

$$2\langle R(z)T(z)\rangle - \langle \mathrm{Tr}(V'(\Phi)\mathcal{T}(z))\rangle + \langle w^\alpha(z)w_\alpha(z)\rangle - \frac{1}{3}G^2\langle T(z)T(z)\rangle = 0\,. \tag{3.6}$$

To arrive at these equations we have repeatedly used the identities in the chiral ring derived in section 2.1. Had we not used those identities, we would have found ambiguities in various terms, due to the fact that cyclic property of the trace is not obeyed. Note that the above equations reduce to the same equations derived in [6] in absence of the graviphoton fields strength. To see this, we have to use the chiral ring equation $G^2 w_\alpha = 0$ in the first equation of (3.4). Finally, the third ingredient in solving for the gauge invariant operators is that the above Ward identities involve two point functions of the gauge invariant operators. In absence of either gravity or the graviphoton field strength these operators factorize in the chiral ring [9]. However, in the presence of these background there is no apriori reason for factorization, in fact the correspondence of the gauge theory with the matrix model and the diagrammatic calculations of [4, 5] imply that these operators do not factorize. Therefore, we need a further set of Ward identities determining the connected two-point functions. For the case of the gravitational background alone, this was done in [6], and it was shown there that the corrections to the gauge invariant operator T is precisely that of genus one correction to the resolvent of the matrix model. We will repeat the same analysis for the case where the graviphoton field strength is turned on. In section 3.1 we set up the anomaly equations which determine the connected two point functions and in section 3.2 we will solve for the one point functions of the gauge invariant operators, to genus one.

Before we proceed, we note that the equations of (3.4) simplify if we perform the following field redefinition

$$R(z) \to R(z) + \frac{1}{6}G^2 T(z)\,. \tag{3.7}$$

This field redefinition shifts all moments in the generating functional, and it is therefore a generalization of the field redefinition noted in [6], which removed the genus zero contribution of the gravitational correction to the superpotential. In fact, in section 3.2 we will see that this field redefinition does the same job in the general case. Using this redefinition in (3.4) we obtain

$$\begin{aligned}
\langle R(z)R(z)\rangle - I(z)\langle R(z))\rangle &= 0\\
2\langle R(z)w_\alpha(z)\rangle - I(z)\langle w_\alpha(z)\rangle &= 0\\
2\langle R(z)T(z)\rangle - I(z)\langle T(z)\rangle + \langle w^\alpha(z)w_\alpha(z)\rangle &= 0\,.
\end{aligned} \tag{3.8}$$

In obtaining these equations we have used the chiral ring identities as well as (3.4). Here the integral operator $I(z)$ is given by

$$I(z)A(z) = \frac{1}{2\pi i}\oint_{C_z} dy \frac{V'(y)A(y)}{y-z}\,, \tag{3.9}$$

with the contour C_z encircling z and ∞. Note that if A is equal to $\mathcal{R}$, ρ_α or $\mathcal{T}$, the integral operator reduces to

$$I(z)A(z) = V'(\Phi)A(z)\,. \tag{3.10}$$

With the field redefinition above, the Ward identities reduce exactly to those with no gravitational or graviphoton bacground, in fact the first equation of (3.8) is identical to the equation for the matrix model resolvent.

3.1 Connected two point functions

Following the same method used in [6] we will derive a set of equations for the connected two point functions. We see that, to solve for the one point functions from (3.8), it is sufficient to determine the connected two-point functions evaluated at coincident points in the z plane. However, it is more convenient to determine the connected two point functions at two different points in the complex plane say z and w, for example $\langle R(z)T(w)\rangle$. This turns out be useful as we can impose conditions on the connected two point function, such as their contour integrals around various branch cuts in z and w plane vanish separately, which enable us to solve the corresponding generalized Konishi anomaly equations completely.

We will briefly review the method used in [6] using the example of the two point function $\langle R(z)T(w)\rangle$. Consider the infinitesimal transformation which is local in superspace coordinates $(x^\mu, \theta, \bar{\theta})$,

$$\delta\Phi_{ij} = \mathcal{R}_{ij}(z)T(w)\,. \tag{3.11}$$

The jacobian of this transformation has two pieces

$$\frac{\delta(\delta\Phi_{ji})}{\delta\Phi_{kl}} = \frac{\delta\mathcal{R}_{ji}(z)}{\delta\Phi_{kl}}T(w) + \sum_m \mathcal{R}_{ji}(z)\mathcal{T}_{mk}(w)\mathcal{T}_{lm}(w) \tag{3.12}$$

The first term in the equation above together with the variation of the classical superpotential gives rise to

$$\begin{aligned}
&\langle (R(z)R(z) - \mathrm{Tr}(V'(\Phi)\mathcal{R}(z)))T(w)\rangle = \\
&\quad = \langle (R(z)R(z) - \mathrm{Tr}(V'(\Phi)\mathcal{R}(z)))\rangle\langle T(w)\rangle + 2\langle R(z)\rangle\langle R(z)T(w)\rangle_c - \\
&\quad\quad - \langle \mathrm{Tr}(V'(\Phi)\mathcal{R}(z)))T(w)\rangle_c + \langle R(z)R(z)T(w)\rangle_c\,,
\end{aligned} \tag{3.13}$$

where the subscript c denotes completely connected 2- or 3-point functions. The first term on the right hand side vanishes when we use the first equation of (3.8). From the second term in the jacobian, when combined with the anomaly (3.2), (3.3), we obtain the following single trace contribution

$$-\frac{1}{3}G^2\langle tr(\mathcal{R}(z)\mathcal{T}(w)\mathcal{T}(w))\rangle = -\frac{1}{3}G^2\partial_w\frac{\langle R(z)\rangle - \langle R(w)\rangle}{z-w} \equiv -\frac{1}{3}G^2\partial_w R(z,w)\,. \tag{3.14}$$

Here we have introduced the notation $A(z,w) = \langle A(z) - A(w)\rangle/(z-w)$ for A equal to R, w_α and T. Note that the field redefinition of (3.7) does not affect the single trace contribution, as it already comes with order G^2. The field redefinition introduces a correction of order G^4 for the above single trace quantity, which vanishes in the chiral ring. Combining (3.13), (3.14) and the first equation of (3.8), one obtains the following equation for the connected correlation functions:

$$(2\langle R(z)\rangle - I(z))\langle R(z)T(w)\rangle_c + \langle R(z)R(z)T(w)\rangle_c - \frac{1}{3}G^2\partial_w R(z,w) = 0\,. \tag{3.15}$$

Using estimates as the ones performed in [6], shows that the completely connected three-point functions vanish. We will not need these estimates when we discussing the solution for all genera, as will be seen in the next section. The method to obtain the other relevant connected correlation functions is similar. We need to consider a general variation of the form

$$\delta\Phi_{ij} = A_{ij}(z)B(w)\,, \tag{3.16}$$

where A can be $\mathcal{R}$, ρ_α or $\mathcal{T}$ and B can be R, w_α or T. The resulting generalized Konishi anomaly equations equations can be derived in the same way as above and are written as the following matrix equation.

$$\begin{bmatrix} M(z) & 2\langle T(z)\rangle & 2\langle w^{\alpha}(z)\rangle \\ 0 & M(z) & 0 \\ 0 & 2\langle w_{\alpha}(z)\rangle & M(z) \end{bmatrix} \begin{bmatrix} \langle T(z)T(w)\rangle_c & \langle T(z)R(w)\rangle_c & \langle T(z)w_{\beta}(w)\rangle_c \\ \langle R(z)T(w)\rangle_c & \langle R(z)R(w)\rangle_c & \langle R(z)w_{\beta}(w)\rangle_c \\ \langle w_{\alpha}(z)T(w)\rangle_c & \langle w_{\alpha}(z)R(w)\rangle_c & \langle w_{\alpha}(z)w_{\beta}(w)\rangle_c \end{bmatrix} =$$

$$= \partial_w \begin{bmatrix} \frac{1}{3}G^2 T(z,w) & \frac{1}{3}G^2 R(z,w) & \frac{1}{3}G^2 w_{\beta}(z,w) \\ \frac{1}{3}G^2 R(z,w) & 16F^2 R(z,w) & -8\,(F\cdot G)_{\beta}\,R(z,w) \\ \frac{1}{3}G^2 w_{\alpha}(z,w) & -8\,(F\cdot G)_{\alpha}\,R(z,w) & Q_{\alpha\beta} \end{bmatrix}. \tag{3.17}$$

Where we have introduced the operators $M(z) = (2\langle R(z)\rangle - I(z))$ and $Q_{\alpha\beta} = \frac{5}{3}G^2\epsilon_{\alpha\beta}$ $R(z,w) - 8(F\cdot G)_{\alpha} w_{\beta}(z,w)$, here the second term is also proportional to $\epsilon_{\alpha\beta}$ using the chiral ring equation (2.20). To obtain such equations we have used the chiral ring relations extensively. We have also dropped all connected three-point functions. The order at which they occur can be inferred using the estimates of [6]: they either vanish in the chiral ring or occur at a higher order. For genus one we are interested in the solution at order G^2, F^2 or $(F\cdot G)$. The equation in (3.17) is of the form

$$\mathcal{M}(z)N(z,w) = \partial_w K(z,w) \tag{3.18}$$

with $\mathcal{M}(z)$ the first matrix operator appearing on the left hand side of eq. (3.17), $N(z,w)$ and $K(z,w)$ satisfy $N(z,w) = N^t(w,z)$ and $K(z,w) = K^t(w,z)$. The non-trivial consistency condition (integrability condition) is then given by

$$(\partial_w K(z,w))\mathcal{M}^t(w) = \mathcal{M}(z)\partial_z K(z,w)\,. \tag{3.19}$$

Using the methods of [6], it can be shown that the above integrability condition is satisfied. The existence of solutions for the connected two point functions, equation (3.17), is guaranteed by the fulfillment of the integrability conditions (3.19). However, as familiar also in matrix models, these solutions suffer from ambiguities, in the form of a finite set of parameters. These ambiguities will be fixed by the physical requirement that the contour integrals around the branch cuts of the connected two point-functions, both in the z and w planes, vanish separately. The reason for this is that the following operator equations hold:

$$\frac{1}{2\pi i}\int_{C_i} dz R(z) = S_i\,, \qquad \frac{1}{2\pi i}\int_{C_i} dw T(w) = N_i\,, \qquad \frac{1}{2\pi i}\int_{C_i} dz w_{\alpha}(z) = w_{\alpha\ i}\,, \tag{3.20}$$

where S_i is the chiral superfield whose lowest component is the gaugino bilinear in the i-th gauge group factor in the broken phase $\mathrm{U}(N) \to \prod_{i=1}^{n}\mathrm{U}(N_i)$ and $w_{\alpha\ i}$ is the U(1) chiral gauge superfield of the $\mathrm{U}(N_i)$ subgroup. Since these fields are background fields, in the connected correlation functions the contour integrals around the branch cuts must vanish. This requirement makes the solutions of the equations of (3.17) unique.

We now write the solution of all connected two-point functions in terms of a single function. Consider the equation for the correlation function $\langle R(z)R(w)\rangle_c$ from the matrix equation (3.17)

$$M\langle R(z)R(w)\rangle_c - 16F^2\partial_w R(z,w) = 0\,. \tag{3.21}$$

The above equation is a linear equation with an inhomogeneous term which is proportional to F^2. Let the solution be given by

$$\langle R(z)R(w)\rangle_c = -16F^2 H^{(1)}(z,w)\,, \tag{3.22}$$

where $H^{(1)}(z,w)$ solves the following equation[5]

$$MH^{(1)}(z,w) + \partial_w R(z,w) = 0\,, \tag{3.23}$$

here the superscript in H refers to the fact that we are working at genus one. It is possible to define such a function, as the inversion of operator M is unambiguous. In [6] it was shown that the function $H(z,w)$ is symmetric in z and w. We now illustrate how the connected two point function $\langle T(z)T(w)\rangle_c$, can be expressed in terms of the function $H^{(1)}(z,w)$. From (3.17), the equation satisfied by this correlator is given by

$$M(z)\langle T(z)T(w)\rangle_c + 2\langle T(z)\rangle\langle R(z)T(w)\rangle_c + 2\langle w^\alpha(z)\rangle\langle w_\alpha(z)T(w)\rangle_c = \frac{G^2}{3}\partial_w T(z,w)\,. \tag{3.24}$$

Now let us define the following operators:

$$D = N_i\frac{\partial}{\partial S_i}\,, \qquad \delta_\alpha = w_{\alpha\ i}\frac{\partial}{\partial S_i}\,. \tag{3.25}$$

Applying the operator $(D+\delta^2/2)$ on equation (3.23) we see that it reduces to (3.24) if one uses the relations $\langle T(z)\rangle = (D+\frac{1}{2}\delta^2)\langle R(z)\rangle$, and $\langle w_\alpha(z)\rangle = \delta_\alpha\langle R(z)\rangle$. Though these relations are valid only to the zeroth order, it is possible to use them here since corrections occur at higher order than G^2 in (3.24). Therefore, using the uniqueness of solutions of the equations involving the operator M, we find

$$\langle T(z)T(w)\rangle_c = -\frac{G^2}{3}(D+\frac{1}{2}\delta^2)H^{(1)}\,. \tag{3.26}$$

We can find all other two point functions in a similar manner, at genus one order they are given by

$$\begin{aligned}
\langle R(z)R(w)\rangle_c &= -16F^2H^{(1)}\,, & \langle R(z)T(w)\rangle_c &= -\tfrac{1}{3}G^2H^{(1)}\,,\\
\langle w_\alpha(z)w_\beta(w)\rangle_c &= \left(-\frac{5}{3}G^2H^{(1)}+8F^2DH^{(1)}\right)\epsilon_{\alpha\beta}\,, & \langle R(z)w_\alpha(w)\rangle_c &= 8(F\cdot G)_\alpha H^{(1)}\,,\\
\langle T(z)w_\alpha(w)\rangle_c &= -\frac{1}{3}G^2\delta_\alpha H^{(1)}\,, & \langle T(z)T(w)\rangle_c &= -\frac{1}{3}G^2\left(D+\frac{1}{2}\delta^2\right)H^{(1)}\,.
\end{aligned} \tag{3.27}$$

3.2 One point functions at genus one

To solve for the corrections to the one point functions we first expand the one point functions as

$$\langle R(z)\rangle = R^{(0)}(z) + R^{(1)}(z)\,, \quad \langle w_\alpha(z)\rangle = w_\alpha^{(0)}(z) + w_\alpha^{(1)}(z)\,, \quad \langle T(z)\rangle = T^{(0)}(z) + T^{(1)}(z)\,, \tag{3.28}$$

[5]This definition of H differs from the one used in [6] by a sign.

where the terms with the superscript 0 denote the zeroth order contribution in F^2, G^2 and $(F \cdot G)$ and terms with the superscript 1 denote the first order contribution. Substituting the above expansions in (3.8) we obtain the following equations for the genus one contributions

$$\begin{aligned}
\Big(2R^{(0)}(z) - I(z)\Big)\, R^{(1)}(z) + 16F^2 H^{(1)}(z,z) &= 0 \\
\Big(2R^{(0)}(z) - I(z)\Big)\, w^{(1)}_\alpha(z) - 16(F\cdot G)_\alpha H^{(1)}(z,z) + 2R^{(1)}(z) w^{(0)}_\alpha(z) &= 0 \\
\Big(2R^{(0)}(z) - I(z)\Big)\, T^{(1)}(z) + 4G^2 H^{(1)}(z,z) - \\
-16F^2 D H^{(1)}(z,z) + 2w^{(0)\,\alpha}(z) w^{(1)}_\alpha(z) + 2R^{(1)}(z) T^{(0)}(z) &= 0\,.
\end{aligned} \tag{3.29}$$

All the above equations are linear in the one point functions with different inhomogeneous terms. The linear operator is $M = (2R^{(0)} - I(z))$, therefore we consider the equation

$$(2R^{(0)}(z) - I(z))\Omega^{(1)}(z) = H^{(1)}(z,z)\,. \tag{3.30}$$

From the definition of the operator $I(z)$ the solution of this equation is given by

$$\Omega^{(1)}(z) = \frac{H^{(1)}(z,z) + c^{(1)}(z)}{2R_0(z) - V'(z)}\,, \tag{3.31}$$

here $c^{(1)}$ is the finite ambiguity in the solution, a polynomial of degree $n-2$. This ambiguity is again fixed by the physical requirement that the contour integral of $\Omega^{(1)}$, which is proportional to the genus one correction to any one one-point of interest, vanishes around branch cuts. This requirement ensures that operator equations (3.20) are valid and the background fields S_i, N_i and $w_{\alpha i}$ do not receive any G^2, F^2 or $(F \cdot G)$ corrections. Using these inputs, the genus one corrections to the one-point functions of interest are given by

$$\begin{aligned}
R^{(1)}(z) &= 16F^2 \Omega^{(1)}(z) \\
w^{(1)}_\alpha(z) &= -16(F\cdot G)_\alpha \Omega^{(1)}(z) \\
T^{(1)}(z) &= 4G^2 \Omega^{(1)}(z) - 16F^2 D\Omega^{(1)}(z)\,.
\end{aligned} \tag{3.32}$$

At this juncture it is worthwhile to point out the difference in the results had we not used the field redefinition in (3.7). We would, in fact, be left with genus zero contributions, in addition to the corrections found in (3.32). This is seen as follows: without the field definition there will be additional terms proportional to G^2, multiplying products of one point functions at the zeroth order. For example, in the last equation of (3.4) there is a term proportional to $G^2(T^{(0)}(z))^2$ which, since it goes like N^2 , it represents a genuine genus 0 contribution,

4. Solution at all genera

In this section we obtain the complete solution for the one point functions R, w_α and T for all genera. From (3.8) we see that we need the connected two point functions $\langle RR\rangle_c, \langle Rw_\alpha\rangle_c, \langle RT\rangle_c$ and $\langle w^\alpha w_\alpha\rangle_c$ to solve for the one point functions. Our strategy in this section is to first obtain equations for generating functionals for any arbitrary correlator using the generalized Konishi anomaly. This can be done by introducing sources coupled to the operators of interest. The equations constraining the generating functionals turn out to be a set of integro-differential equations. From these we solve for the relevant two-point functions which in turn enables us to obtain the one point-functions of interest. We then compare the results with the matrix model. In the appendix we demonstrate that the integro-partial differential equations constraining the generating functional are consistent and provide the details of the solution for all connected two point functions.

4.1 Generating functionals for connected correlators.

To obtain equations for the generating functionals for connected correlators we extend the method used to obtain equations for the connected two point functions. Consider the following generating functional for the operators of our interest

$$\langle Z\rangle = \left\langle \exp\left[\int dw\,\left(j_R(w)R(w) + j_w^\alpha(w)w_\alpha(w) + j_T(w)T(w)\right)\right]\right\rangle . \tag{4.1}$$

The generating functional is a function of three variables j_R, j_w^α, j_T.There are three equations which constrain Z which are obtained by considering the following three variations

$$\delta\Phi_{ji} = \mathcal{R}_{ji}(z)Z\,, \qquad \delta\Phi_{ji} = \eta^\alpha \rho_{\alpha ji}(z)Z\,, \qquad \delta\Phi_{ji} = \mathcal{T}_{ji}(z)Z\,, \tag{4.2}$$

here η^α is an arbitrary spinor. We will now derive the constraint imposed by the Konishi anomaly for the first variation. The derivation proceeds along the lines followed to derive the equations for the connected two-point functions, discussed in section 3.1. The jacobian of the first variation in (4.2) is given by

$$\frac{\delta(\delta\Phi_{ji})}{\delta\Phi_{kl}} = \frac{\delta\mathcal{R}_{ji}}{\delta\Phi_{kl}}Z + \mathcal{R}_{ji}(z)\frac{\delta Z}{\delta\Phi_{kl}}\,, \tag{4.3}$$

where

$$\begin{aligned}\frac{1}{Z}\frac{\delta Z}{\delta\Phi_{kl}} = \sum_m \int dw\Big(& j_R(w)\mathcal{R}_{mk}(w)\mathcal{T}_{lm}(w) + j_w^\alpha(w)w_{\alpha mk}(w)\mathcal{T}_{lm}(w) + \\ &+ j_T(w)\mathcal{T}_{mk}(w)\mathcal{T}_{lm}(w)\Big)\,.\end{aligned} \tag{4.4}$$

Note that the expression contains terms similar to the jacobians in (3.12). Therefore, we can use the anomaly equations constraining the two-point functions for obtaining the equations constraining the generating functional. The first term of the jacobian in (4.4) gives rise to a term which is the product of the first anomaly equation in (3.8) times Z, while the second term of the jacobian in (4.4) gives rise to inhomogeneous single trace terms similar to those in (3.14). Using these considerations, the anomaly equation for the above variation can be shown to reduce to

$$\begin{aligned}&\langle R(z)R(z)Z\rangle - I(z)\langle R(z)Z\rangle - \\ -&\left\langle \int dw\left(16F^2 j_R\partial_w R(z,w) + \frac{G^2}{3}j_T\partial_w R(z,w) - 8j_w^\alpha(F\cdot G)_\alpha \partial_w R(z,w)\right)Z\right\rangle = 0\,,\end{aligned} \tag{4.5}$$

where $R(z,w) = (R(z) - R(w))/(z-w)$, $T(z,w) = (T(z) - T(w))/(z-w)$. We can write the above equation as an integro-differential equation by introducing functional derivatives with respect to the sources on the generating functional. We define

$$\begin{aligned}
\partial_R^z &= \frac{\delta}{\delta j_R(z)}\,, \qquad \partial_T^z = \frac{\delta}{\delta j_T(z)}\,, \qquad \partial_\alpha^z = \frac{\delta}{\delta j_w^\alpha(z)}\,, \\
O_R^{(z,w)} &= \partial_w \left(\frac{1}{z-w} \left[\frac{\delta}{\delta j_R(z)} - \frac{\delta}{\delta j_R(w)} \right] \right), \\
O_T^{(z,w)} &= \partial_w \left(\frac{1}{z-w} \left[\frac{\delta}{\delta j_T(z)} - \frac{\delta}{\delta j_T(w)} \right] \right), \\
O_\alpha^{(z,w)} &= \partial_w \left(\frac{1}{z-w} \left[\frac{\delta}{\delta j_w^\alpha(z)} - \frac{\delta}{\delta j_w^\alpha(w)} \right] \right).
\end{aligned} \tag{4.6}$$

Now writing (4.5) in terms of these derivatives gives

$$\left[\partial_R^2 - I(z)\partial_R - \int dw \left(16F^2 j_R O_R + \frac{G^2}{3} j_T O_R - 8 j_w^\alpha (F\cdot G)_\alpha O_R \right) \right] Z = 0 \tag{4.7}$$

here we have suppressed the superscripts z, w in the derivatives for clarity of notation. The generalized Konishi anomaly constraints from the other two variations in (4.2) are given by

$$\begin{aligned}
&\left[2\partial_R \partial_\alpha - I(z)\partial_\alpha + \right.\\
&\left. + \int dw \left(\left(8 j_R (F\cdot G)_\alpha + \frac{5}{3} G^2 j_{w\alpha} \right) O_R - \left(\frac{G^2}{3} j_T + 8 j_w^\beta (F\cdot G)_\beta \right) O_\alpha \right) \right] Z = 0 \\
&\left[2\partial_R \partial_T - I(z)\partial_T + \partial^\alpha \partial_\alpha - \frac{G^2}{3} \int dw \left(j_R O_R + j_T O_T + j_w^\alpha O_\alpha \right) \right] Z = 0
\end{aligned} \tag{4.8}$$

The above equations form a set of closed integro-partial differential equations which determine Z

For the connected two point functions of our interest it is sufficient to consider the following ansatz for the generating functional

$$Z = \exp(M) = \exp\left(M_R(j_R) + j_T M_T(j_R) + j_w^\alpha M_\alpha(j_R) + \frac{1}{2} j_w^\alpha j_w^\beta M_{\alpha\beta}(j_R) \right). \tag{4.9}$$

Here the product $j_T M_T$ is understood to mean $\int dz j_T(z) M_T(z)$, similarly for other products in the above expression. In the appendix we will generalize this to include all two point functions. Note that for the various cumulants in the ansatz we have allowed an arbitrary dependence of j_R while we allow only a finite set of moments in j_T and j_w^α. As we will see subsequently, it is possible that higher moments in j_R are non vanishing, but higher moments in the other currents truncate in the chiral ring. An indication of this is clear form the nature of the two point functions at genus one in (3.27). Only the $\langle RR\rangle_c$ correlator is a function of F^2 alone, all the others involve powers of G, thus they will truncate at some order in the chiral ring. With this ansatz the connected two-point functions of interest are given by

$$\langle R(z)R(w)\rangle_c = \partial_R^z\partial_R^w M_R(j_R)\,, \qquad \langle R(z)w_\alpha(w)\rangle_c = \partial_R^z M_\alpha(w,j_R)\,,$$
$$\langle R(z)T(w)\rangle_c = \partial_R^z M_T(w,j_R)\,, \qquad \langle w_\alpha(z)w_\beta(w)\rangle_c = M_{\alpha\beta}(z,w,j_R)\,. \tag{4.10}$$

The three equations of (4.7) and (4.8) in terms of the cumulant generating functional M become

$$\begin{aligned}(\partial_R M)^2 + \Big[\partial_R^2 - I(z)\partial_R -& \\ -\int dw\left(16F^2 j_R O_R + \frac{G^2}{3}j_T O_R - 8j_w^\alpha (F\cdot G)_\alpha O_R\right)\Big]M &= 0\end{aligned} \tag{4.11}$$

$$\begin{aligned}2\partial_R M\partial_\alpha M + \Big[2\partial_R\partial_\alpha - I(z)\partial_\alpha +& \\ +\int dw\left(\left(8j_R(F\cdot G)_\alpha + \frac{5}{3}G^2 j_{w\alpha}\right)O_R - (\frac{G^2}{3}j_T + 8j_w^\beta(F\cdot G)_\beta)O_\alpha\right)\Big]M &= 0\end{aligned} \tag{4.12}$$

$$\begin{aligned}2\partial_R M\partial_T M + \partial^\alpha M\partial_\alpha M +& \\ +\left[2\partial_R\partial_T - I(z)\partial_T + \hat\partial^\alpha\hat\partial_\alpha - \frac{G^2}{3}\int dw\,(j_R O_R + j_T O_T + j_w^\alpha O_\alpha)\right]M &= 0\,.\end{aligned} \tag{4.13}$$

As M_R is only a function of j_R, the equation determining M_R can be obtained from (4.11), by setting $j_T = j_\alpha = 0$. This is given by

$$(\partial_R M_R)^2 + \left[\partial_R^2 - I(z)\partial_R - 16F^2\int dw j_R O_R\right]M_R = 0\,. \tag{4.14}$$

We will see in subsection 4.4 that the above equation is identical to the generating functional equation for the resolvent of the matrix model. In fact the F^2 expansion of the above equation can be identified with the $1/\hat N^2$ expansion of the equation of the resolvent of the matrix model. We now find the solutions of the connected two-point functions of interest, the analysis is similar to that of the genus one case in section 3. In fact, the solution is a direct generalization of that case. To obtain the connected two point function $\langle RR\rangle_c$ differentiate (4.11) by ∂_R^w, where w refers to another point in the complex plane and set $j_T = j_\alpha = 0$. We obtain the following equation

$$\begin{aligned}(2\partial_R^z M_R - I(z))\partial_R^z\partial_R^w M_R + (\partial_R^z)^2\partial_R^w M_R - 16F^2 O_R^{(z,w)}M_R -& \\ -16F^2\int dw' j_R O_R^{(z,w')}\partial_R^w M_R = 0\,.&\end{aligned} \tag{4.15}$$

This forms a basic equation out of which the solutions for the other cumulant generating functions in (4.9) will be constructed. The equation is linear in $\partial_R^z\partial_R^w M_R$ with an inhomogeneous proportional to F^2, therefore the solution is proportional to F^2. Let the solution of of (4.15) be given by

$$\langle R(z)R(w)\rangle_c = \partial_R^z\partial_R^w M_R = -16F^2 H(z,w)\,, \tag{4.16}$$

where $H(z,w)$ is the solution of the following equation

$$\begin{aligned}(2\partial_R^z M_R - I(z))H(z,w) + \partial_R^z H(z,w) +& \\ +O_R^{(z,w)}M_R - 16F^2\int dw' j_R\partial_{w'}\left(\frac{H(z,w) - H(w',w)}{z-w'}\right) = 0\,.&\end{aligned} \tag{4.17}$$

By this definition $H(z,w)$ is symmetric in z and w. We have assumed that the solutions of these equations are unique. An argument in favour of this is as follows. For the lowest order in the genus expansion (setting $j_R = 0$, and $\partial^z_R H(z,w) = 0$) these equations reduce to the ones which were studied in [6]. It was shown there that, by demanding the vanishing of the integrals of the the various connected two-point functions around the branch cuts both in the z and w plane, the solution is unique. The equation in (4.17) is a generalization of those equations. One can envisage a generalization of those arguments for these equations, proving that the solution of (4.17) is unique. To obtain the connected two point function $\langle R(z) w_\alpha(w)\rangle_c$ we differentiate (4.11) with respect to $\hat{\partial}^w_\alpha$ and then set $j_T = j_\alpha = 0$, to obtain

$$(2\partial^z_R M_R - I(z))\partial^z_R M_\alpha(w) + \partial^z_R\partial^z_R M_\alpha(w) + 8(F\cdot G)_\alpha O^{(z,w)}_R M_R - \\ -16F^2 \int dw' j_R O^{(z,w')}_R M_\alpha(w) = 0\,. \quad (4.18)$$

Again comparing (4.17) and (4.18) we obtain

$$\langle R(z) w_\alpha(w)\rangle_c = \partial^z_R M_\alpha(w) = 8(F\cdot G)_\alpha H(z,w)\,. \quad (4.19)$$

At this point one might wonder if the equation for the above correlator obtained by differentiating (4.12) by ∂^z_R will reduce to (4.18). In the appendix we will see this is indeed the case and that the set of integro-differential equations in (4.11), (4.12) and (4.13) is in fact consistent. In order to obtain the correlator $\langle R(z) T(w)\rangle_c$ we differentiate (4.11) by ∂^w_T and then set $j_T = j_\alpha = 0$, we obtain

$$(2\partial^z_R M_R - I(z))\partial^z_R M_T(w) + \partial^z_R\partial^z_R M_T(w) - \frac{G^2}{3} O^{(z,w)}_R M_T(w) - \\ -16F^2 \int dw' j_R O^{(z,w')}_R M_T(w) = 0\,. \quad (4.20)$$

Comparing (4.20) and (4.17) yields

$$\langle R(z) T(w)\rangle_c = \partial^z_R M_T(w) = -\frac{G^2}{3} H(z,w)\,. \quad (4.21)$$

Now differentiate (4.12) with respect to ∂^z_β and then set $j_\alpha, j_T = 0$ to obtain the connected correlator $\langle w_\alpha(z) w_\beta(w)\rangle_c$. We get

$$(2\partial^z_R M_R - I(z))M_{\alpha\beta}(z,w) + 2\partial^z_R M_\beta(w) M_\alpha(z) + \\ +2\partial^z_R M_{\alpha\beta}(z,w) - 8(F\cdot G)_\alpha \int dw' j_R O^{(z,w')}_R M_\beta(w) - \frac{5}{3} G^2 \epsilon_{\alpha\beta} O_R(w,z) M_R - \\ -8(F\cdot G)_\beta \partial_w \left(\frac{M_\alpha(z) - M_\alpha(w)}{z-w}\right) = 0 \quad (4.22)$$

The last term in the above equation can be written as

$$-8(F\cdot G)_\beta \partial_w \left(\frac{M_\alpha(z) - M_\alpha(w)}{z-w}\right) = 8\epsilon_{\alpha\beta} F^2 \partial_w \left(\frac{M_T(z) - M_T(w)}{z-w}\right), \\ = 8\epsilon_{\alpha\beta} F^2 D O^{(z,w)}_R M_R\,. \quad (4.23)$$

In the last equality we have used the relation, $T = (D + 1/2\delta^2)R$ which is valid at the zeroth order and the chiral ring relation (2.11) and (2.20). The second term in (4.22) contains $\partial_R^z M_\beta(w)$. Substituting (4.19) for this we and using the equation (2.11) we see that this term vanishes. Using (4.19) we see that term containing the integral in (4.22) is proportional to $(F \cdot G)_\alpha (F \cdot G)_\beta$, which also vanishes using (2.12). For clarity we write down (4.22) after dropping these terms

$$\begin{aligned}&(2\partial_R^z M_R - I(z))M_{\alpha\beta}(z,w) + 2\partial_R^z M_{\alpha\beta}(z,w) - \\ &\quad -\epsilon_{\alpha\beta}\frac{5}{3}G^2 O_R(w,z)M_R + 8\epsilon_{\alpha\beta}F^2 D O_R^{(z,w)} M_R = 0 \end{aligned} \tag{4.24}$$

There are two inhomogeneous terms in the above equation which motivates the following ansatz

$$M_{\alpha\beta}(z,w) = \epsilon_{\alpha\beta}\left(-\frac{5}{3}G^2 H(z,w) + 8F^2 DH(z,w)\right). \tag{4.25}$$

With this ansatz the term containing the connected three-point function vanishes, because it contains the derivative ∂_R^z which contains an extra factor of F^2. This is seen as follows: (4.17) is identical to the corresponding matrix model equation for the connected two point function of the resolvent, from a t'Hooft counting analysis, the connected three point function is down by a factor of $1/\hat{N}^2$, which in the gauge theory implies that there is an extra factor of F^2 since the F^2 expansion of (4.14) and (4.17) is identical to the $1/\hat{N}^2$ expansion of the matrix model. Comparing (4.17), we see that (4.25) solves the (4.24), as with this ansatz the connected three point function in (4.17) also vanishes, since it occurs with an extra factor of F^2.

4.2 Solutions for the one point functions.

Having obtained the required connected two point functions we can now solve for the corrections to the one point functions. The analysis is identical to the genus one case. We first expand the one point functions about the zeroth order as

$$\langle R(z)\rangle = R^{(0)}(z) + \tilde{R}(z), \qquad \langle w_\alpha(z)\rangle = w_\alpha^{(0)}(z) + \tilde{w}_\alpha(z), \qquad \langle T(z)\rangle = T^{(0)}(z) + \tilde{T}(z), \tag{4.26}$$

here $\tilde{R}$, $\tilde{w}$ and $\tilde{T}$ denote corrections to the zeroth order solution. Now substituting the above expansion in (3.8) we obtain the same equations as (3.29) but with $H^{(1)}(z,z)$ replaced with the full connected two point function $H(z,z)$, which is the solution of (4.17). These are given below

$$\begin{aligned}\left(2R^{(0)} - I(z)\right)\tilde{R}(z) - 16F^2 H(z,z) &= 0, \\ \left(2R^{(0)} - I(z)\right)\tilde{w}_\alpha(z) + 16(F\cdot G)_\alpha H(z,z) + 2\tilde{R}w_\alpha^{(0)} &= 0, \\ \left(2R^{(0)} - I(z)\right)\tilde{T}(z) - 4G^2 H(z,z) + 16F^2 DH(z,z) + 2\tilde{w}^\alpha w_\alpha^{(0)} + 2T^{(0)}\tilde{R} &= 0. \end{aligned} \tag{4.27}$$

The corrections are given by

$$\begin{aligned}\tilde{R}(z) &= 16F^2\Omega(z), \qquad \tilde{w}_\alpha = -16(F\cdot G)_\alpha\Omega(z), \\ \tilde{T}(z) &= 4G^2\Omega(z) - 16F^2 D\Omega(z), \end{aligned} \tag{4.28}$$

where

$$\Omega(z) = \frac{1}{2R^{(0)} - V'(z)}(H(z,z) + c(z))\,, \tag{4.29}$$

where $c(z)$ is a polynomial of degree $n-2$ and is uniquely determined by the requirement that the contour integrals of $\Omega(z)$ around each branch cut vanishes. Note that the last term of the second equation in (4.27) vanishes, because after substituting the solution for $\tilde{R}$, we see that that term is proportional to $F^2 w_\alpha$, which is trivial in the chiral ring. In the section 4.4 we show that this is exactly the answer obtained from the matrix model.

4.3 Shift invariance of the anomaly equations

In this section we show that we can assemble all the equations for the generating functions given in (4.11), (4.12) and (4.13) into one superfield equation by introducing the auxiliary fermionic coordinate ψ_α. We first assemble the loop variables as

$$\mathcal{R}(z,\psi) = R(z) + \psi^\alpha w_\alpha(z) - \frac{1}{2}\psi^2 T(z)\,. \tag{4.30}$$

Note that with this notation the generalized Konishi anomaly equations (3.8) is given by

$$\langle \mathcal{R}^2(z,\psi)\rangle - I(z)\langle \mathcal{R}(z,\psi)\rangle = 0\,. \tag{4.31}$$

These generalized Konishi anomaly equations are the same as in the case of $\mathcal{N}=1$ gauge theories in flat space with no graviphoton background. In that case, the connected two-point functions in (4.31) vanish, and this shift symmetry of the equations implied that the superpotenial could be written as the integral [9]

$$W_{\mathrm{eff}} = \int d^2\psi \mathcal{F}_p\left(S_i + \psi^\alpha w_{\alpha i} - \frac{1}{2}\psi^2 N_i\right)\,. \tag{4.32}$$

We would like to demonstrate that this is also true for the case of $\mathcal{N}=1$ theories studied here.

We first show that the equations for the connected correlators can be written as a supermultiplet in the ψ space. The background fields, the $\mathcal{N}=1$ Weyl multiplet and the $\mathcal{N}=1$ spin 3/2 multiplet can be assembled as

$$\mathcal{H}_{\alpha\beta}(\psi) = F_{\alpha\beta} - \frac{1}{2}\psi^\gamma G_{\alpha\beta\gamma}\,. \tag{4.33}$$

To write the generating functional in the superspace ψ we introduce the multiplet of the sources as follows

$$\mathcal{J}(z,\psi) = J_T + \psi^\alpha j_\alpha - \frac{\psi^2}{2} j_R\,. \tag{4.34}$$

Then the generating functional can be written as

$$Z = \exp\left(\int d^2\psi dw \mathcal{J}(w,\psi)\mathcal{R}(w,\psi)\right)\,, \tag{4.35}$$

here we have normalized $\int d^2\psi\psi^2 = -2$. To obtain the various correlators from the generating functional we introduce the following derivative in the supermultiplet space

$$\partial^z_{\mathcal{R}} = \partial^z_R + \psi^\alpha\partial^z_\alpha - \frac{\psi^2}{2}\partial^z_T\,. \tag{4.36}$$

With this notation the set of equations (4.11), (4.12) and (4.13) reduce to

$$\Big[(\partial^z_{\mathcal{R}})^2 - I(z)\partial^z_{\mathcal{R}} - \frac{2}{3}\hat{D}^2(\mathcal{H}^2)\int dw \mathcal{J}(w,\psi)O^{(z,w)}_{\mathcal{R}} + \\ +8\int dw \hat{D}^2(\mathcal{H}^{\alpha\beta}\mathcal{J}(w,\psi))\mathcal{H}_{\alpha\beta}O^{(z,w)}_{\mathcal{R}}\Big] Z = 0\,, \tag{4.37}$$

where

$$\hat{D}^2 = \epsilon^{\alpha\beta}\frac{\partial}{\partial\psi^\beta}\frac{\partial}{\partial\psi^\alpha}\,. \tag{4.38}$$

In (4.37) $O_{\mathcal{R}}$ is defined using the super derivative of (4.36). In deriving (4.37) we have used the identities (2.8), (2.11) and (2.20). This implies that the solutions of the loop equations can be written in a shift invariant way. It is easy to verify this from the solutions we have found in the previous subsection. The corrections to the one point functions can be written as

$$\begin{aligned}\tilde{\mathcal{R}}(\psi,z) &= 16\mathcal{H}^2\Omega\left(S_i + \psi^\alpha w_{\alpha i} - \frac{1}{2}\psi^2 N_i, \mathcal{H}^2\right), \\ &= 16F^2\Omega(S_i,F^2) - 16\psi^\alpha(F\cdot G)_\alpha\Omega(S_i,0) - \\ &\quad -\frac{1}{2}\psi^2\left(4G^2\Omega(S_i,0) - 16F^2 D\Omega(S_i,0)\right).\end{aligned} \tag{4.39}$$

To obtain the expansion in the second line we have used the chiral ring relation $F^2 w_{\alpha i} = 0$ and $2F_{\alpha\beta}N_i + G_{\alpha\beta\gamma}w^\alpha_i = 0$. Here the dependence of F^2 in the last two terms are set to zero as the expansion in F truncates in the chiral ring for these terms. Since the solutions to the correlators in $\mathcal{R}$ can be written as a supermulitplet in ψ, the corrections to the super potential must be written as

$$\tilde{W}_{\text{eff}} = \int d^2\psi \mathcal{H}^2\mathcal{F}\left(S_i + \psi^\alpha w_{\alpha i} - \frac{1}{2}\psi^2 N_i\right). \tag{4.40}$$

This motivates the completion of the gravitational corrections to the superpotential to as

$$\tilde{W}_{\text{eff}} = \int d^2\psi \mathcal{H}^{2g}\mathcal{F}_g\left(S_i + \psi^\alpha w_{\alpha i} - \frac{1}{2}\psi^2 N_i\right) \tag{4.41}$$

which is in agreement with what is obtained from the closed string duality with ψ playing the role of the second $\mathcal{N}=2$ superspace coordinate. However because of the identities in the chiral ring all terms for $g>1$ are trivial form $\mathcal{N}=1$ point of view. Nevertheless, the generating functional $\langle R(z)\rangle$ sees the complete genus expansion and is identified with the matrix model genus expansion as we will see in the next subsection.

4.4 Comparison with the matrix model results.

Consider a hermitean matrix model with an action given by

$$S = \frac{\hat{N}}{g_m}\sum_k \frac{g_k}{k}\,\mathrm{Tr}\,M^k \equiv \frac{\hat{N}}{g_m}V\,, \tag{4.42}$$

where M is a hermitean $\hat{N} \times \hat{N}$ matrix. The basic loop equations for the resolvent is given by

$$\langle \Omega_m(z)\Omega_m(z)\rangle - I(z)\langle \Omega_m(z)\rangle = 0 \tag{4.43}$$

here Ω_m is the matrix model resolvent given by

$$\Omega_m(z) = \frac{g_m}{\hat{N}} \operatorname{Tr}\left(\frac{1}{z-M}\right). \tag{4.44}$$

We now obtain the loop equations satisfied by the variation

$$\delta M_{ji} = \Omega_{m\,ji} \exp\left(\int dw J(w)\Omega_m(w)\right) = \Omega_{m\,ji} Z_m \tag{4.45}$$

here Z_m is the generating functional for the n-point functions for the resolvent of the matrix model. The loop equations for the variation in (4.45) is given by

$$\left\langle \left[\Omega_m^2(z) - I(z)\Omega_m(z) + \left(\frac{g_m}{\hat{N}}\right)^2 \int dw J \partial_w \left(\frac{\Omega_m(z)-\Omega_m(w)}{z-w}\right)\right] Z_m \right\rangle = 0\,. \tag{4.46}$$

Writing this loop equation by introduction functional derivatives in the current J we get

$$\left[\partial_J^2 - I(z)\partial_J + \left(\frac{g_m}{\hat{N}}\right)^2 \int dw J O_J\right] Z_m = 0\,. \tag{4.47}$$

To obtain equations for the connected correlators we introduce the cumulant generating functional $Z_m = \exp(M_m)$. The cumulant generating functional satisfies the following equation

$$(\partial_J M_m)^2 + \left[\partial_J^2 - I(z)\partial_J + \left(\frac{g_m}{\hat{N}}\right)^2 \int dw J O_J\right] M_m = 0\,. \tag{4.48}$$

This equation is identical to the equation (4.14) which is satisfied by cumulant generating functional M_R of the gauge theory. The F^2 expansion in the gauge theory is analogous to the $1/\hat{N}^2$ expansion in the matrix model. The equation for the connected two point function is obtained by differentiating the above equation with ∂_J^w, which is given by

$$\begin{aligned}(2\partial_J^z M_m - I(z))\partial_J^z\partial_J^w M_m + \partial_J^z\partial_J^z\partial_J^w M_m + \left(\frac{g_m}{\hat{N}}\right)^2 O_J^{(z,w)} + \\ + \left(\frac{g_m}{\hat{N}}\right)^2 \int dw' J O_J^{(z,w')}\partial_J^w M_m = 0\,.\end{aligned} \tag{4.49}$$

By comparison with (4.17) we see that the solution of the connected two point function of the matrix model resolvent is given by

$$\langle \Omega_m(z)\Omega_m(w)\rangle = \partial_J^z\partial_J^w M_m = \left(\frac{g_m}{\hat{N}}\right)^2 H(z,w)\,. \tag{4.50}$$

Here again we have used the fact that the inversion of the operator $(2\partial_J^z M_m - I(z))$ is unambiguous if the ambiguity is fixed by demanding contour integrals of the two point function around the branch cuts vanish. Note that in (4.49) using a t'Hooft counting analysis one finds that the contribution of a l loop planar Feynman graph to the three point function $\partial_J^z \partial_J^z \partial_J^w M_m$ is proportional to $(g_m/\hat{N})^{(4+l)}$. Therefore its contribution is subleading compared to the other two terms, this and the fact that (4.48) and (4.14) are identical was used in the gauge theory analysis to drop certain three point functions in the chiral ring. To find the the solution of the one point function of the resolvent we expand $\Omega_m(z) = \Omega_m^{(0}(z) + \tilde{\Omega}_m(z)$, and substitute it in (4.46). $\Omega_m^{(0)}$ is the contribution of the planar graphs to the resolvent. The correction $\tilde{\Omega}_m(z)$ satisfies the following equation

$$((2\Omega_m^{(0)}(z) - I(z))\tilde{\Omega}_m(z) + \left(\frac{g_m}{\hat{N}}\right)^2 H(z,z) = 0\,. \tag{4.51}$$

Demanding that the contour integral of $\tilde{\Omega}_m(z)$ around the branch cuts are vanishing ensures that the correction is identical to (4.28), therefore proving the all-orders matching between gauge theory and matrix model correlators claimed in the introduction.

5. Conclusions

In this paper we have analyzed the issue of gravitational effective F-terms, resulting from the integration of a massive adjoint scalar with arbitrary tree level superpotential and coupled to $\mathcal{N} = 1$ super Yang-Mills and supergravity, with the non-trivial, non-standard $F_{\alpha\beta}$ background. The main results we obtained are, firstly, the determination of the chiral ring relations in the presence of both $F_{\alpha\beta}$ and $G_{\alpha\beta\gamma}$, thereby extending previous analysis of [6]. Secondly, the proof that the gauge theory one-point function $\langle R(z)\rangle$ coincides to all orders with the matrix model one-point function $\langle \Omega(z)\rangle$, provided one identifies the expansion parameters on the two sides. As mentioned in the introduction, this is equivalent to the statement that the gauge theory correlators $\langle \mathrm{Tr}\, W^2 \Phi^k\rangle$ and the matrix model ones, $\langle \mathrm{Tr}\, M^k\rangle$, are identical to all orders in a genus expansion, after the appropriate identification of the expansion parameters.

Concerning terms in the gauge theory low energy action, we have also argued that the gravitational effective superpotential can be written in the form $\int d^2\psi \mathcal{H}^{2g} \mathcal{F}_g(S + \psi^\alpha w_\alpha - \frac{1}{2}\psi^2 N)$, as expected from [8, 4, 5], with $\mathcal{F}_g$ related to a genus-g matrix model partition function. We have however observed that, due to the chiral ring relations of section 2, the above superpotential turns out to be trivial (i.e. $\bar{D}$-exact) from the $\mathcal{N} = 1$ point of view for $g \geq 2$. Of course, the equality between gauge theory and matrix model correlators noted above, survives the chiral ring relations, since it involves only powers of F^2 on the gauge theory side.

From the technical point of view, we have followed a strategy based on generalized anomaly equations and chiral ring relations, as first exploited in [9] for the non-gravitational case and then adapted to the case of super Yang-Mills theory coupled to supergravity for genus one in [6]. As already stressed, one of the points of the present work was to find the correct chiral ring relations in the $\mathcal{N} = 1$ gauge theory coupled to supergravity, with a background $F_{\alpha\beta}$: whereas we cannot claim we have a "microscopic" gauge theoretic derivation of them, we find the fact that our assumptions agree, for gauge invariant operators, with the relations we obtain on the dual, closed string side, rather remarkable.

As for some open problems, it should be interesting to extend our approach to other matter representations and to the other classical (or exceptional) groups, extending the analysis of [18, 19] which did not involve gravity. Also, remaining in the context discussed here, on the gauge theory side we have used the resolvent-like operators, $R(z)$, $w_\alpha(z)$ and $T(z)$, but only their contour integrals around the branch cuts in the z-plane (which in the semiclassical limit become the critical points of the superpotential), S_i, $w_{\alpha i}$ and N_i are identified with the closed string vector multiplets. It would be interesting to see if there is a more general correspondence, which, on the gauge theory side involves higher powers of Φ.

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A. Other two point correlators and integrability conditions

For completeness we derive the connected two point functions $\langle T(z)T(w)\rangle_c$ and $\langle w_\alpha(z) T(w)\rangle_c$. These are not used in evaluating the full one-point functions, but serve to demonstrate the consistency of the constraints on the generating functional given by the equations (4.11), (4.12) and (4.13). For convenience we define the following moments

$$\begin{aligned} \partial_T^w \partial_\alpha^z M|_{j_T=j_w^\alpha=0} &= M_{\alpha T}(z,w) = \langle w_\alpha(z)T(w)\rangle_c\,, \\ \partial_T^w \partial_T^z M|_{j_T=j_w^\alpha=0} &= M_{TT}(z,w) = \langle T(z)T(w)\rangle_c\,. \end{aligned} \tag{A.1}$$

We proceed as in section 4.1, to determine the correlator $\langle w_\alpha(z)T(w)$. We differentiate (4.12) by ∂_T^w and then set $j_T = j_\alpha = 0$ which gives

$$\begin{aligned} &(2\partial_R^z M_R - I(z))M_{\alpha T}(z,w) + 2\partial_R^z M_T(w)M_\alpha(z) + 2\partial_R^z M_{\alpha T}(z,w) + \\ &\quad +8(F\cdot G)_\alpha \int dw' j_R O_R^{(z,w')} M_T(w) - \frac{G^2}{3}\partial_z\left(\frac{M_\alpha(z)-M_\alpha(w)}{z-w}\right) = 0\,. \end{aligned} \tag{A.2}$$

First note that the term containing the integral drops out as it involves a $\partial_R^z M_T(w)$ which, using (4.21), is proportional to $G^2(F\cdot G)$ and thus is trivial in the chiral ring. The solution of the above equation can also be related to the function $H(z,w)$, and this can be seen as follows: differentiate (4.17) by δ_α, to obtain

$$\begin{aligned} &(2\partial_R^z M_R - I(z))\delta_\alpha H(z,w) + 2\delta_\alpha \partial_R^z M_R H(z,w) + \partial_R^z \delta_\alpha H(z,w) - \\ &\quad -16F^2 \int dw' j_R \partial_z\left(\frac{\delta_\alpha H(z,w) - \delta_\alpha H(w',w)}{z-w'}\right) + \delta_\alpha O_R^{(z,w)} M_R = 0\,. \end{aligned} \tag{A.3}$$

Now we note that for the one point functions we have the following relations

$$\delta_\alpha \partial_R^z M_R = M_\alpha(z)\,, \qquad \delta_\alpha O_R^{(z,w)} M_R = M_\alpha(z,w)\,. \tag{A.4}$$

These relations are valid at the zeroth order, but it is sufficient for our purpose as we will consider a solution which is proportional to G^2 . Therefore the higher order corrections are trivial in the chiral ring. By comparing (A.2) and (A.3) we see that the ansatz

$$M_{T\alpha}(w,z) = -\frac{G^2}{3}\delta_\alpha H(z,w) \tag{A.5}$$

solves (A.2), as it reduces to (A.3). To show this one uses (A.4), the fact that the term containing the integral vanishes in the chiral ring for both the equations and that the three point functions in (A.2) and (A.3) vanish with the ansatz of (A.5), as they come with a higher power of F^2. Finally, to obtain the correlator $\langle T(z)T(w)\rangle_c$ we take the derivative of (4.13) with respect to ∂_T^w and set $j_T = j_\alpha = 0$, giving the following equation

$$\begin{aligned}(2\partial_R^z M_R - I(z))M_{TT}(w,z) + 2\partial_T^w\partial_R^z M_R M_T(z)+\\ +2M^\alpha(z)M_{\alpha T}(z,w) + 2\partial_R^z M_{TT}(z,w)+\\ +\partial_T^w\hat{\partial}^{z\alpha}\hat{\partial}_\alpha^z M|_{j_T,j_\alpha=0} - \frac{G^2}{3}\int dw j_R O_R^{(w',z)}M_T(z) - \frac{G^2}{3}\partial_w\left(\frac{M_T(z)-M_T(w)}{z-w}\right) = 0\,.\end{aligned} \tag{A.6}$$

On substituting the value of $O_R^{(w',z)}M_T(z)$ from (4.21) in the term containing the integral, we see that it is proportional to G^4 and therefore trivial in the chiral ring. We can also substitute the solutions of $\partial_T^w\partial_R^z M_R$ and M_{TT} obtained in (4.21) and (A.5) in the above equations. With this substitution it is easy to see that the solution of (A.9) equation can be written in terms of $H(z,w)$, differentiating (4.17) by the operator $D+\delta^2/2$. We obtain

$$\begin{aligned}(2\partial_R^z M_R - I(z))\left(D+\frac{1}{2}\delta^2\right)H(z,w) + 2\left(D+\frac{1}{2}\delta^2\right)\partial_R^z M_R H(z,w)+\\ +2\delta^\alpha\partial_R^z M_R\delta_\alpha H(z,w) + \left(D+\frac{1}{2}\delta^2\right)\partial_R^z H(z,w) + \left(D+\frac{1}{2}\delta^2\right)O_R^{(z,w)}M_R-\\ -16F^2\left(D+\frac{1}{2}\delta^2\right)\int dw' j_R\partial_{w'}\left(\frac{H(z,w)-H(w',w)}{z-w'}\right) = 0\,.\end{aligned} \tag{A.7}$$

We also have the following relation at the zeroth order

$$\left(D+\frac{1}{2}\delta^2\right)\partial_R^z M_R = M_T(z)\,. \tag{A.8}$$

Again it is sufficient to use the relation at the zeroth order, since we will be interested in a solution which is proportional to G^2, therefore higher order corrections to the above relation vanish in the chiral ring. From comparing (A.6) and (A.7) and using the relations (A.4) and (A.8), we see that the following ansatz satisfies (A.6)

$$M_{TT}(w,z) = -\frac{G^2}{3}\left(D+\frac{1}{2}\delta^2\right)H(z,w)\,. \tag{A.9}$$

Note that, with this ansatz, the terms containing the integral and the three-point functions in both (A.6) and (A.7) vanish, so that they reduce to the same equation.

To verify that the equations constraining the generating functionals are consistent we obtain the correlators $\langle R(z)w_\alpha(w)\rangle_c$, $\langle R(z)T(w)\rangle_c$, $\langle w_\alpha(z)T(w)\rangle_c$ by a different route and show that they lead to the same results as discussed earlier. It is possible to obtain these correlators by a different route, as partial derivatives commute and it is not obvious that the results for the correlators (4.11), (4.12) and (4.13) will be the same.

First consider the correlator $\langle R(z)w_\alpha(w)\rangle_c$, in (4.18) we had obtained it by differentiating (4.11) by ∂_α^w, we could also obtain the same correlator by differentiating (4.12) by ∂_R^w. Here we verify that we get the same result. On performing the differentiation and setting $j_T = j_\alpha = 0$ we obtain

$$(2\partial_R M - I(z))\partial_R^w M_\alpha(z) + 2\partial_R^w\partial_R^z M_R M_\alpha(z) + \\ +2\partial_R^w\partial_R^z M_\alpha(z) + \int dw' 8 j_R (F\cdot G)_\alpha O_R^{(z,w')} M_\alpha(w) + 8(F\cdot G)_\alpha O_R^{(z,w)} M_R = 0\,. \quad (A.10)$$

The following ansatz solves the equation (4.19)

$$\partial_R^w M_\alpha(z) = 8(F.G)_\alpha H(z,w)\,. \quad (A.11)$$

To see this, note that with this ansatz the term containing $\partial_R^w\partial_R^z M_R M_\alpha(z)$ is trivial in the chiral ring as it is proportional to $F^2 w_\alpha$, the term containing the integral also vanishes. Therefore comparing (4.17) and (A.10) we see that the above ansatz satisfies the latter equation. The solution is consistent with the one obtained in (4.19) as $H(z,w)$ is a symmetric function in z and w. Consider the two point function $\langle RT\rangle_c$, we now obtain this correlator by differentiating (4.13) with ∂_R^w and verify that the result is consistent with (4.21). On performing the differentiation and setting $j_T = j_\alpha = 0$ we obtain

$$(2\partial_R - I(z))\partial_R^w M_T(z) + 2\partial_R^w\partial_R^z M_R M_T(z) + 2\partial_R^w M^\alpha(z) M_\alpha(z) + \\ +2\partial_R^w\partial_R^z M_T^z + \partial_R^w M_\alpha^\alpha(z,z) - \frac{G^2}{3}\int dw' j_R O_R^{(z,w')}\partial_w M_R - \frac{G^2}{3} O_R^{(z,w)} M_R = 0\,. \quad (A.12)$$

The combination of the second and third terms in the above equation is trivial in the chiral ring as shown below

$$2\partial_R^w\partial_R^z M_R M_T(z) + 2\partial_R^w M^\alpha(z) M_\alpha(z) = 2(16F^2 M_T(z) - 8(F\cdot G)_\alpha M^\alpha(z))H(z,w) = 0\,. \quad (A.13)$$

Here we have used (2.8). Now comparing (A.12) and (4.17), it is clear that the following ansatz solves (A.12)

$$\partial_R^w M_T(z) = -\frac{G^2}{3} H(z,w)\,. \quad (A.14)$$

The terms containing the integral and the connected three point functions are trivial in the chiral ring, with the ansatz in (A.14) for both (A.12) and (4.17) as they come with a higher power of F^2. This solution is consistent with the one obtained in (4.21) because $H(z,w)$ is a symmetric function. Finally we consider the two-point function $\langle Tw_\alpha\rangle$, we obtain this by differentiating (4.13) by ∂_α^w and setting $j_T = j_\alpha = 0$

$$(2\partial_R^z M_R - I(z))M_{\alpha T}(w,z) + 2\partial_R^z M_\alpha(w) M_T(z) + 2M_\alpha{}^\beta(z,w)M_\beta(z) + \\ +2\partial_R M_{T\alpha}(z,w) + \partial_\alpha^w\partial^{\beta z}\partial_\beta M|_{j_T=j_\alpha=0} - \frac{G^2}{3}\int dw' j_R O_R^{(z,w')} M_\alpha(w) - \\ -\frac{G^2}{3}\partial_w\left(\frac{M_\alpha(z) - M_\alpha(w)}{z-w}\right) = 0\,. \quad (A.15)$$

Again the second and third term of the above equation can be further simplified in the chiral ring, by substituting the correlators from (4.19) and (4.25)

$$
\begin{aligned}
&2\partial_R^z M_\alpha(w)M_T(z) + 2M_\alpha^{\ \beta}(z,w)M_\beta(z) = \\
&\quad = 2\left(8(F\cdot G)_\alpha H(z,w)M_T(z) + \frac{5}{3}G^2 H(z,w)M_\alpha(z) - 8F^2 DH(z,w)M_\alpha(z)\right) \\
&\quad = -\frac{2}{3}G^2 M_\alpha(z)H(z,w)\,.
\end{aligned} \tag{A.16}
$$

Here we have used the chiral ring equations (2.8) and (2.11) Now comparing (A.15) and (A.3) we see that the the solution can be written as

$$
M_{\alpha T}(z,w) = -\frac{G^2}{3}\delta_\alpha H(w,z)\,. \tag{A.17}
$$

The terms containing the three point functions and the integrals in (A.15) and (A.3) vanish with the above solution. This concludes the proof of the integrability of the constraints on the generating functional.

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